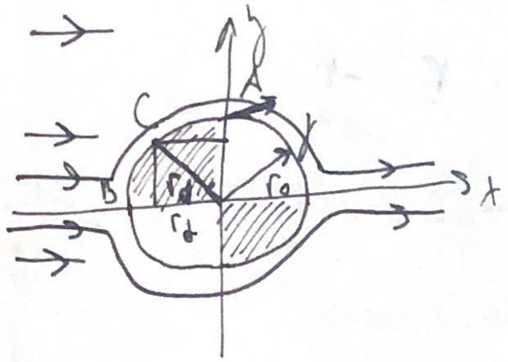
Пример обтекания кругового цилиндра



$$\Delta\psi = 0; \quad \psi\big|_\gamma = 0$$

$$\begin{cases} \Delta\psi = 0 \\ \psi\big|_\gamma = 0 \\ V_x = \dfrac{\partial\psi}{\partial y} \underset{x^2+y^2\to\infty}{\equiv} V_0 \qquad x^2+y^2\to\infty \\ V_y = \dfrac{\partial\psi}{\partial x} \underset{x^2+y^2\to\infty}{=} 0 \qquad V(V_0, 0) \end{cases}$$

$$\Psi = \psi - V_0 y \Rightarrow$$

$$\Rightarrow \begin{cases} \Delta\Psi = 0 \\ \Psi\big|_\gamma = -V_0 y \\ \dfrac{\partial\Psi}{\partial y} = 0 \\ \dfrac{\partial\Psi}{\partial x} = 0 \end{cases} \leftarrow x^2+y^2\to\infty$$

Перейдём к полярным коорд-ам $\Rightarrow \psi(r,\varphi)$

Ищем реш-е в виде $\psi = R(r)\Phi(\varphi)$

$$\Delta\Psi = \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\psi\right)\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\varphi^2} = 0$$

$\Psi = R(r)\Phi(\varphi)$ – ищем реш-е в виде

$$\frac{1}{r}\left(r\cdot R'(r)\Phi(\varphi)\right)'_r + \frac{1}{r^2}R(r)\Phi''(\varphi) = 0 \;\Big|\cdot r^2$$

$$r\left(R'(r)\cdot\Phi(\varphi) + rR''(r)\cdot\Phi(\varphi)\right) + R(r)\cdot\Phi''(\varphi) = 0$$

$$r\frac{R'(r)}{R(r)} + r^2\frac{R''(r)}{R(r)} = -\frac{\Phi''(\varphi)}{\Phi(\varphi)} = \lambda = \text{const}$$

$$\Psi = R(r)\cdot\Phi(\varphi), \quad \Phi(\varphi) = \Phi(\varphi+2\pi)$$

1) $\Phi''(\varphi) + \lambda\Phi(\varphi) = 0$

$k^2 + \lambda = 0, \quad k_{1,2} = \pm\sqrt{-\lambda}$

1а) $\lambda > 0 \Rightarrow \Phi = A\cos(\sqrt{\lambda}\varphi) + B\sin(\sqrt{\lambda}\varphi)$

1б) $\lambda < 0 \quad \Phi = C_1 e^{k_1\varphi} + C_2 e^{k_2\varphi}$ – усл-е периодичности не вып-ся $\Rightarrow$ $\Rightarrow$ не подх.

1) $\lambda > 0, \ \Phi = A\cos(\sqrt{\lambda}\varphi) + B\sin(\sqrt{\lambda}\varphi)$

$\Phi(\varphi) = \Phi(\varphi + 2\pi)$

$$A\cos(\sqrt{\lambda}\varphi) + B\sin(\sqrt{\lambda}\varphi) = A\cos(\sqrt{\lambda}\varphi + \sqrt{\lambda}2\pi) + B\sin(\sqrt{\lambda}\varphi + \sqrt{\lambda}2\pi) \Rightarrow \sqrt{\lambda} = n, \ \lambda = n^2$$

$$\Phi(\varphi)=\sum_{n=0}^{\infty} A_n(\cos n\varphi)+B_n\sin(n\varphi)$$

2) $$\frac{rR'(r)}{R}+\frac{r^2R''(r)}{R}=\lambda$$

$$r(rR')'=n^2R$$

$$R=r^n,\quad R=r^{-n}$$

$$\Psi=\sum_{n=0}^{\infty}\left(\frac{r_0}{r}\right)^n\left(A_n\cos(n\varphi)+B_n\sin n\varphi\right)$$

$$\Psi\Big|_{y=r=r_0}=-V_0y=-V_0r_0\sin\varphi$$

$$\boxed{\Psi\Big|_{r=r_0}=\sum_{n=0}^{\infty}\left(\frac{r_0}{r_0}\right)^n\left(A_n\cos(n\varphi)+B_n\sin(n\varphi)\right)=-V_0r_0\sin\varphi}$$

$$A_n=0,\quad n=0,1,\dots$$

$$B_n=0,\quad n=2,\dots$$

$$B_1=-Vr$$

$$\Psi=\left(\frac{r_0}{r}\right)(-V_0\cdot r_0)\sin\varphi=-\frac{V_0r_0^2}{r}\sin\varphi=$$

$$=-\frac{V_0r_0^2}{r^2}(r\sin\varphi)$$

В ДСК: $$\Psi=-\frac{V_0r_0^2}{x^2+y^2}+y$$

$$\Psi=\psi-V_0y$$

$$\psi=V_0y\left(1-\frac{r_0^2}{x^2+y^2}\right)$$

$$V_x=\frac{\partial\psi}{\partial y};\quad V_y=-\frac{\partial\psi}{\partial x}$$

$$V_x=V_0\left(1-\frac{r^2}{x^2+y^2}\right)+V_0y\left(\frac{r_0^2\cdot 2y}{(x^2+y^2)^2}\right)=$$

$$=V_0\left(1-\frac{r_0^2}{(x^2+y^2)^2}\left(\frac{x^2+y^2}{1}-2y^2\right)\right)=$$

$$=V_0\left(1-\frac{r_0^2(x^2-y^2)}{(x^2+y^2)^2}\right)$$

$$V_y=-\frac{\partial\psi}{\partial x}=-\left(\frac{V_0yr_0^2\cdot 2x}{(x^2+y^2)^2}\right)=-\frac{V_0r_0^2 2xy}{(x^2+y^2)^2}$$

(см. рис. в начале)

(A) $x=0,\ y=r_0$

$$V_x=V_0\left(1-\frac{r_0^2(-r_0)^2}{r_0^4}\right)=V_0(1+1)=2V_0$$

$$V_y=0$$

(B) $x=-r_0;\ y=0\Rightarrow V_y=0$

$$V_x=V_0\left(1-\frac{r_0^2r_0^2}{r_0^4}\right)=0$$

(C) $$V_y=\frac{-V_0r_0^2-2\frac{r_0}{\sqrt2}\left(-\frac{r_0}{\sqrt2}\right)}{(r_0)^4}=V_0$$

$$\nabla\left(\frac{p}{\rho}+\frac{V^2}{2}\right)=0;\quad p+\rho\frac{V^2}{2}=\text{const}$$

для выдел. струй

$$p_\infty+\rho\frac{V_0^2}{2}=p+\rho\frac{V^2}{2}$$

$$p-p_\infty=\rho\left(\frac{V_0^2-V^2}{2}\right)=\rho\frac{V_0^2}{2}\left(1-\left(\frac{V}{V_0}\right)^2\right);$$

$$\bar p=\frac{p-p_\infty}{\rho\frac{V_0^2}{2}}=1-\left(\frac{V}{V_0}\right)^2;$$

$$\left(\frac{v_x}{v_0}\right)^2 = \left(1 + \frac{r_0^2(y^2-x^2)}{(x^2+y^2)^2}\right)^2 = \left(1 + \frac{y^2-x^2}{x^2+y^2}\right)^2 = \frac{4y^4}{r_0^4} = \frac{4(r_0^2-x^2)^2}{r_0^4} = 4\left(1-\left(\frac{x}{r_0}\right)^2\right)^2;$$

для точки на контуре

$x^2+y^2 = r_0^2$

$$\bar{p} = 1 - 4\left(1-\left(\frac{x}{r_0}\right)^2\right)^2;$$

A: $y=0$; $x=(-r_0) \Rightarrow \bar{p}=1$;

$$p_A - p_\infty = \rho\frac{v_0^2}{2} \Rightarrow p_A = p_\infty + \rho\frac{v_0^2}{2}, \quad p_A > p_\infty$$

B: $y=r_B$; $x=0$; $\Rightarrow \bar{p} = -3$;

$$p_B - p_\infty = -3\rho\frac{v_0^2}{2} \Rightarrow p_B = p_\infty - 3\rho\frac{v_0^2}{2}, \quad p_B < p_\infty$$